Exercise 49

Use logarithmic differentiation to find the derivative of the function.

$$y = (\tan x)^{1/x}$$

Solution

Take the natural logarithm of both sides and use the properties of logarithms to simplify the right side.

$$\begin{aligned} \ln y &= \ln(\tan x)^{1/x} \\ &= \left(\frac{1}{x}\right)\ln\tan x \end{aligned}$$

Differentiate both sides with respect to x.

$$\begin{aligned} \frac{d}{dx}(\ln y) &= \frac{d}{dx}\left[\left(\frac{1}{x}\right)\ln\tan x\right] \\ \frac{1}{y}\cdot\frac{d}{dx}(y) &= \left[\frac{d}{dx}\left(\frac{1}{x}\right)\right]\ln\tan x + \left(\frac{1}{x}\right)\left[\frac{d}{dx}(\ln\tan x)\right] \\ \frac{1}{y}\cdot\frac{dy}{dx} &= \left(-\frac{1}{x^2}\right)\ln\tan x + \left(\frac{1}{x}\right)\left[\frac{1}{\tan x}\cdot\frac{d}{dx}(\tan x)\right] \\ \frac{1}{y}\frac{dy}{dx} &= -\frac{\ln\tan x}{x^2} + \left(\frac{1}{x}\right)\left[\frac{\cos x}{\sin x}\cdot(\sec^2 x)\right] \\ \frac{dy}{dx} &= y\left(-\frac{\ln\tan x}{x^2} + \frac{1}{x\sin x\cos x}\right) \\ &= (\tan x)^{1/x}\left(-\frac{\ln\tan x}{x^2} + \frac{1}{x\sin x\cos x}\right) \\ &= (\tan x)^{1/x}\left(-\frac{\sin x\cos x\ln\tan x}{x^2\sin x\cos x} + \frac{x}{x^2\sin x\cos x}\right) \\ &= (\tan x)^{1/x}\left(\frac{x - \sin x\cos x\ln\tan x}{x^2\sin x\cos x}\right) \\ &= \frac{(\tan x)^{1/x}}{x^2\sin x\cos x}(x - \sin x\cos x\ln\tan x) \end{aligned}$$